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DEGREE SEQUENCES OF RANDOM GRAPHS

BÉLA BOLLOBÁS

Department of Pure Mathematics and Mathematical Statistics, Cambridge University, Cambridge, United Kingdom/Mathematics Institute, Aarhus, Denmark

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The paper sets out to investigate the degree sequences $d_1 \geq d_2 \geq \cdots \geq d_n$ of random graphs of order n in which the edges are chosen independently and with the same probability p, $0<p<1$. The main aim is to find small intervals which contain the mth degree, d_m, of almost every (a.e.) graph. It is shown that if m varies arbitrarily in the interval $1 \leq m \leq \frac{1}{2}n$ and $C(n) \to \infty$ arbitrarily slowly, then there is an interval of length $C(n)\,(n/\log(n/m))^{1/2}$ which contains the mth highest degree, d_m, of a.e. graph. If m grows slowly with n (in particular if $m=1$, that is $d_m = d_1$ is the maximal degree), the centre of this interval is given by a simple expression.

Other results concern the jumps $d_m - d_{m+1}$ in the degree sequence of a.e. graph, the repeated values in the degree sequence, the number of edges that can be covered by m vertices, and the connectivity of large subgraphs.

The results of the paper are essentially best possible.

0. Introduction

The study of random graphs was initiated by Erdös [7] in order to show the existence of a graph of large chromatic number and large girth. Since then random graphs have often been used to attack traditional problems of graph theory (see [2, 5, 6, 8, 10, 12, 20, 21]).

Erdös and Rényi [9] investigated random graphs for their own sake. Their main aim was to determine the number of edges a random graph should have to ensure that almost every graph has a given property. The striking discovery of Erdös and Rényi was that many properties appear rather suddenly. Since then many authors have discussed this phenomenon for different properties (see [4, 10, 11, 12, 14, 16, 17, 18]). In particular, in the model $\mathcal{G}(n, P(\text{edge})=p)$ the clique number is almost determined for most values of n if n is large enough.

The aim of this paper is to continue the second line of investigation. We cannot show that, say, the maximal degree Δ of a random graph is almost uniquely determined (for it is not, of course), but we shall estimate the different members of the degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ and the jumps $d_m - d_{m+1}$. Erdös and Wilson [12] have already announced an estimate of $\Delta = d_1$ and claimed that almost every graph has a unique maximal degree. As particular cases of our results, we obtain finer estimates of d_1 and $d_1 - d_2$.

We shall use the terminology and some of the notation of [1]. Throughout the paper $0<p<1$ is fixed and $q=1-p$. We write $c, c_1, c_2, \ldots$ for positive absolute constants or for positive values depending only on p. Different occurrences of c may, of course, denote different constants. We shall always work in the probability space $\mathcal{G}(n, P(\text{edge})=p)$. This is a discrete probability space consisting of all $2^{\binom{n}{2}}$ graphs with n fixed and labelled vertices, in which the probability of a graph with M edges is $p^M q^{\binom{n}{2}-M}$. Equivalently, in our labelled random graph the edges are chosen independently and with the same probability p. Note that every graph invariant (maximal degree, diameter, connectivity, chromatic number, etc.) is a random variable defined on $\mathcal{G}(n, P(\text{edge})=p)$. We shall often say that a certain statement holds for *almost every* (a.e.) graph. This means that as $n\to\infty$ the probability of the set of graphs for which the assertion fails tends to 0.

In Section 1 we list some standard formulae from probability theory. The heart of the paper is Section 2, where we give fairly precise estimates for various terms of the degree sequence $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$. Let us state here a minor result of Section 2: almost every graph is such that its maximal degree $\Delta = d_1$ satisfies

$$\left| d_1 - pn - (2pqn\log n)^{1/2} + \left(\frac{pqn}{8\log n}\right)^{1/2} \log\log n \right| \leqslant \left(\frac{n}{\log n}\right)^{1/2} \log\log\log n.$$

(This corrects an error in [12].)

In fact, the interval to which d_m is confined in a.e. graph decreases as m increases.

After the results of Section 2 it is straightforward to estimate the jumps $d_m - d_{m+1}$ and the multiplicities of the degrees that occur in a.e. graph. This will be done in Section 3. The last section contains some applications of our main results. In particular, we estimate the maximal number of edges that can be covered by m vertices, and we prove the rather surprising fact that in a.e. graph G we can find vertices $x_1, x_2, \ldots$ such that omitting these vertices consecutively we increase the connectivity: $\kappa(G) < \kappa(G) - \{x_1\}) < \kappa(G - \{x_1, x_2\}) < \cdots$.

1. Probability-theoretic preliminaries

In probabilistic graph theory one often needs good approximations of the binomial distribution. Fortunately there is no shortage of such approximations; in fact, we shall hardly need Littlewood's very precise estimate [15] for the probability in the tail of the binomial distribution In this section we shall list for future use some consequences of the classical DeMoivre–Laplace formulae (see Feller [13, pp. 164–179] or Rényi [19, pp. 204–210]).

Here and in what follows we use Landau's notation $O(f(n))$ for a term that, when divided by $f(n)$, remains bounded as $n\to\infty$. Similarly $o(f(n))$ denotes a term that, when divided by $f(n)$, tends to 0 as $n\to\infty$. Thus $O(1)$ denotes a bounded term and $o(1)$ a term tending to 0.

Denote the normal density function by $\varphi(x)$ and the normal distribution function by $\Phi(x)$, that is put

$$\varphi(x)=\frac{1}{\sqrt{(2\pi)}}\,\mathrm{e}^{-x^2/2}$$

and

$$\Phi(x)=\int_{-\infty}^{x}\varphi(t)\,\mathrm{d}t=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{x}\mathrm{e}^{-t^2/2}\,\mathrm{d}t.$$

If $x>0$ and l is a natural number then

$$\frac{1}{(\sqrt{(2\pi)}}\frac{1}{x}\mathrm{e}^{-x^2/2}\left(1+\sum_{m=1}^{2l+1}(-1)^m\frac{1\cdot 3\cdot\ldots\cdot(2m-1)}{x^{2m}}\right)<1-\Phi(x)$$
$$<\frac{1}{\sqrt{(2\pi)}}\frac{1}{x}\mathrm{e}^{-x^2/2}\left(1+\sum_{m=1}^{2l}(-1)^m\frac{1\cdot 3\cdot\ldots\cdot(2m-1)}{x^{2m}}\right). \qquad (1)$$

In particular, if $x\to\infty$, then

$$1-\Phi(x)\sim\frac{1}{\sqrt{(2\pi)}}\frac{1}{x}\mathrm{e}^{-x^2/2}, \qquad (1')$$

where, as in the rest of the paper, ~ means that the percentage difference of the two sides tends to 0 as $n\to\infty$.

Given natural numbers k and n, $k\leq n$, denote by $b(n;k)$ the kth term of the binomial distribution:

$$b(n;k)=\tbinom{n}{k}p^kq^{n-k}.$$

If $k=pn+x(pqn)^{1/2}=pn+\mathrm{o}(n^{2/3})$, that is $x=\mathrm{o}(n^{1/6})$, then we have

$$b(n;k)\sim\frac{1}{\sqrt{(2\pi)}}(pqn)^{-1/2}\mathrm{e}^{-x^2/2}. \qquad (2)$$

Furthermore, for natural numbers K and L, $K\leq L\leq n$, write

$$S(n;K,L)=\sum_{k=K}^{L}\tbinom{n}{k}p^kq^{n-k}=\sum_{k=K}^{L}b(n;k).$$

For simplicity, put also $S(n;K)=S(n;K,n)$. Note that $b(n;k)$, $S(n;K,L)$ and $S(n;K)$ all depend on p as well. However, as p is fixed throughout the paper, we do not indicate this dependence.

If $K=pn+x(pqn)^{1/2}=pn+\mathrm{o}(n^{2/3})$, that is if $x=\mathrm{o}(n^{1/6})$, then

$$S(n;K)\approx 1-\Phi(x). \qquad (3)$$

Also, if $K=pn+\mathrm{o}(n^{2/3})$ and $L=pn+\mathrm{o}(n^{2/3})$, then

$$S(n;K,L)\sim\Phi(y')-\Phi(x'), \qquad (4)$$

where $x'=(K-\frac{1}{2}-pn)/(pqn)^{1/2}$ and $y'=(L+\frac{1}{2}-pn)/(pqn)^{1/2}$.

It seems to be worth remarking here that formulas (2), (3) and (4) will be used to estimate $b(n-i;k)$, $S(n-i;K)$ and $S(n-i;K,L)$, where i is a fixed integer, usually 1. This is legitimate since in each of these cases, the replacement of n by $n-i$ results in a change that is negligible compared to the accuracy of the main approximation, and is certainly small enough to make no difference to our estimates.

To conclude this section, let us note two consequences of the inequalities of Chebyshev. Both of these inequalities are immediate from first principles. If X is a non-negative random variable with expectation $\mu=E(X)>0$ and $\mu'>0$, then

$$P(X\geq\mu')\leq\mu/\mu'. \tag{5}$$

If, furthermore, X has variance $\sigma^2=E((X-\mu)^2)=E(X^2)-\mu^2$ and $\nu>0$, then

$$P(|X-\mu|\geq\nu)\leq\sigma^2/\nu^2. \tag{6}$$

In particular,

$$P(X=0)\leq\sigma^2/\mu^2. \tag{6'}$$

2. The distribution of degrees

Throughout the paper $d_1\geq d_2\geq\cdots\geq d_n$ will denote the degree sequence of a random graph $G\in\mathcal{G}(n, P(\text{edge})=p)$. In order to estimate a term d_m of the degree sequence, we introduce some random variables on $\mathcal{G}(n, P(\text{edge})=p)$. Given a number K, $0\leq K\leq n-1$, denote by X_K the number of vertices of degree at least K in a random graph. Note that $d_m=d_m(G)\geq K$ iff $X_K=X_K(G)\geq m$. Our estimates of d_m will be based on the following simple assertion.

Lemma 1. *Let* $1\leq K'<K''\leq n-1$. *Put* $\mu'=E(X_{K'})$, $\sigma'^2=E((X_{K'}-\mu')^2)$ *and define* μ'' *and* σ''^2 *analogously. If* m *is an integer satisfying* $\mu''<m\leq\mu'$ *then*

$$P(d_m\geq K'')\leq\min\left\{\frac{\mu''}{m},\frac{\sigma''^2}{(m-\mu'')^2}\right\}$$

and

$$P(d_m<K')\leq\frac{\sigma'^2}{(\mu'-m+1)^2}.$$

Proof. The assertions follow from the Chebyshev inequalities (5) and (6). Thus

$$P(d_m\geq K'')=P(X_{K''}\geq m)\leq\frac{\mu''}{m},$$

$$P(d_m\geq K'')=P(X_{K''}\geq m)=P(X_{K''}-\mu''\geq m-\mu'')\leq\frac{\sigma''^2}{(m-\mu'')^2}$$

and

$$P(d_m<K')=P(X_{K'}\leq m-1)=P(\mu'-X_{K'}\geq\mu'-m+1)\leq\frac{\sigma'^2}{(\mu'-m+1)^2}. \qquad\square$$

Depending on the amount of work we are willing and able to put into estimating $\mu = E(X_K)$ and $E((X_K-\mu)^2)$ for various values of K, we get different estimates of d_m.

Lemma 2. *For every integer K, $0 \leq K \leq n-1$, we have*

$$E(X_K) = nS(n-1; K).$$

Proof. The probability that a given vertex has degree k is

$$\binom{n-1}{k} p^k q^{n-1-k}.$$

since we have $\binom{n-1}{k}$ choices for the k edges. As there are n vertices, the assertion follows. $\square$

In this note we shall be interested in X_K only for $K = pn + x(pqn)^{1/2} = pn + o(n^{2/3})$ and usually we shall even assume that $x \to \infty$ (as $n \to \infty$). Under these assumptions Lemma 2 and relations (1′) and (3) imply

$$E(X_K) \sim \frac{1}{\sqrt{(2\pi)}} \frac{n}{x} e^{-x^2/2}. \tag{7}$$

In particular, if $x = 2(\log n)^{1/2}$ then $E(X_K) = O(1/n)$. Furthermore, keeping the assumption that K is an integer

$$E(X_K) - E(X_{k+1}) = nb(n-1; K) \sim (2\pi pq)^{-1/2} n^{1/2} e^{-x^2/2}.$$

Thus if $x \sim (2 \log n)^{1/2}$, the jump $E(X_K) - E(X_{K+1})$ is only of order $n^{-1/2}$, and if $x \geq (1+\varepsilon)$ $(\log n)^{1/2}$ for some fixed $0 < \varepsilon < \frac{1}{4}$ the jump is still less than $n^{-\varepsilon}$. Consequently as K decreases from $\lfloor pn + 2(pqn \log n)^{1/2} \rfloor$ to $\lfloor pn + (1+\varepsilon)$ $(pqn \log n)^{1/2} \rfloor$, the values taken by $E(X_K)$ increase from less than $1/n$ to more than $n^{1/2-2\varepsilon}$, by jumps less than $n^{-\varepsilon}$. In particular, if $1 \leq m \leq n^{1/2-2\varepsilon}$ then there is an integer $K = pn + O((n \log n)^{1/2})$ such that $m \leq E(X_K) \leq m + n^{-\varepsilon}$.

In fact, a little calculation shows that for every m there is a K such that

$$|m - E(X_K)| \leq cm^{1/2} + 1, \tag{8}$$

where c depends only on p.

Lemma 3. *Let K be an integer, $1 \leq K \leq n-1$. Put $\mu = E(X_K)$ and $\sigma^2 = E((X_K - \mu)^2)$. Then*

$$\sigma^2 \leq \mu + n^2 b(n-2; K-1)^2.$$

Proof. Put

$$Y_K = \binom{X_K}{2},$$

that is denote by Y_K the number of pairs of vertices of degree at least K. As $m^2 = m + 2\binom{m}{2}$ for every non-negative integer m, we have

$$E(X_K^2) = E(X_K) + 2E(Y_K).$$

The expectation of Y_K is easily computed:

$$E(Y_K) = \binom{n}{2}\{pS(n-2; K-1)^2 + qS(n-2; K)^2\}.$$

Indeed, the probability that two given vertices, say a and b, both have degree at least K is

$$P(ab \in E(G) \quad \text{and} \quad d_{G-b}(a) \geq K-1,\ d_{G-a}(b) \geq K-1)$$
$$+ P(ab \notin E(G) \quad \text{and} \quad d_{G-a}(b) \geq K,\ d_{G-b}(a) \geq K).$$

The events $\{ab \in E(G)\}$, $\{d_{G-b}(a) \geq K-1\}$ and $\{d_{G-a}(b) \geq K-1\}$ are independent so the probability of their intersection is the product of their probabilities.

$$P(ab \in E(G))P(d_{G-b}(a) \geq K-1)P(d_{G-a}(b) \geq K-1) = pS(n-2; K-1)^2.$$

The second term is calculated analogously. Hence

$$\begin{aligned} \sigma^2 &= E((X_K - \mu)^2) = E(X_K^2) - \mu^2 \\ &= \mu + n(n-1)\{pS(n-2; K-1)^2 + qS(n-2; K)^2\} - \mu^2 \\ &\leq \mu + n^2\{pS(n-2; K-1)^2 + qS(n-2; K)^2 - S(n-1; K)^2\}. \end{aligned}$$

Note now that

$$S(n-1; K) = pS(n-2; K-1) + qS(n-2; K)$$

and

$$S(n-2; K-1) - S(n-2; K) = \binom{n-2}{K-1}p^{K-1}q^{n-K-1} = b(n-2; K-1),$$

so by the convexity of the function $y = x^2$, we have

$$\sigma^2 \leq \mu + n^2 b(n-2; K-1)^2. \quad \square$$

Lemma 4. *Let* $K = pn + x(pqn)^{1/2} = pn + o(n^{2/3})$. *Put* $\mu = E(X_K)$ *and* $\sigma^2 = E((X_K - \mu)^2)$. *Then*

$$\sigma^2 \leq c\mu.$$

Proof. Since $X_K = X_{\lceil K \rceil}$, we may assume that K is an integer. By Lemma 3

$$\sigma^2 \leq \mu + n^2 b(n-2; K-1)^2.$$

We shall use (7) to estimate μ and (2) to estimate $b(n-2; K-1)$.

Suppose first that $x \geq 1$. Then if n is large,

$$\sigma^2 \leq \mu + n^2 \frac{e^{-x^2}}{\pi npq} \leq \mu + \frac{1}{\sqrt{(3\pi)}}\frac{n}{x}e^{-x^2/2} \leq \mu + \frac{1}{pq}\mu,$$

since $x \leq e^{x^2/2}$. Assume now that $x < 1$. Then if n sufficiently large,

$$\mu \geq \frac{1}{\sqrt{(3\pi)}} \frac{n}{1} e^{-1/2} \geq \tfrac{1}{6} n$$

and

$$n^2 b(n-2; K-1)^2 \leq n^2 \frac{1}{\pi n p q} = \frac{1}{\pi p q} n.$$

Finally, the constant c can be chosen to take care of the small values of n. □

Theorem 5. *Let $K' > K''$ be such that $K' = pn + o(n^{2/3})$ and $K'' = pn + o(n^{2/3})$. Suppose $\mu'' = E(X_{K''}) < m < \mu' = E(X_{K'})$. Then*

$$P(d_m \geq K'') \leq c \min \{\mu''/m, \mu''/(m - \mu'')^2\}$$

and

$$P(d_m < K') \leq c\mu'/(m - \mu')^2.$$

Proof. The assertion is immediate from Lemmas 1 and 4.

Theorem 5 is our main tool in estimating d_m. Now we have to get down to some numerical details. Let us see first what we can say about d_m if m is fixed or grows slowly with n.

Let $f = f(n) = o(\log n)^{1/2})$ and put

$$K_f = pn + (2pqn \log n)^{1/2} - \left(\frac{pqn}{2 \log n}\right)^{1/2} \{\tfrac{1}{2} \log \log n + f + \log (2\pi^{1/2})\}. \tag{9}$$

Then because of relation (7) following Lemma 2 we have

$$\mu = E(X_{K_f}) \sim e^f. \tag{9'}$$

Theorem 6. *Let m be a fixed natural number and let f_1, f_2 be fixed real numbers satisfying $e^{f_1} < m < e^{f_2}$. Then*

$$P(d_m \geq K_{f_1}) \leq \min \{e^{f_1}/m, e^{f_1}/(m - e^{f_1})^2\} + o(1)$$

and

$$P(d_m < K_{f_2}) \leq e^{f_2}/(e^{f_2} - m)^2 + o(1).$$

Proof. The assertions are immediate from Theorem 5 and from the relations $E(X_{f_i}) \to e^{f_i}$, $i = 1, 2$. □

Corollary 7. *If m is fixed and $C(n) \to \infty$ (arbitrarily slowly) then a.e. graph satisfies*

$$K^* + C(n)\left(\frac{n}{\log n}\right)^{1/2} \geq d_1 \geq d_2 \geq \cdots \geq d_m \geq K^* - C(n)\left(\frac{n}{\log n}\right)^{1/2},$$

where

$$K^* = pn + (2pqn \log n)^{1/2} - \left(\frac{pqn}{8 \log n}\right)^{1/2} \log \log n.$$

Of course, for a fixed f we may replace K^* by K_f in the corollary above.

One may think that the interval to which the maximal degree of a.e. graph is confined to is small enough and as $m \to \infty$, d_m will be more free to roam about. As we shall presently see, this is not the case: as $m \to \infty$, d_m is confined to smaller and smaller intervals in a.e. graph. In order to achieve this, we have to make full use of Theorem 5. The following assertion is practically a reformulation of Theorem 5, but it enables us to see our objectives clearly.

Theorem 8. *Suppose $m \to \infty$ and $K' < k < K''$ are such that with $\mu = E(X_K)$, $\mu' = E(X_{K'})$ and $\mu'' = E(X_{K''})$ we have*

$$|\mu - m| \leqslant cm^{1/2}, \tag{10}$$

$$\frac{1}{c} C(n)\mu^{1/2} \leqslant \mu' - \mu \leqslant c\mu \tag{11}$$

and

$$\frac{1}{c} C(n)\mu^{1/2} \leqslant \mu - \mu'' \leqslant c\mu, \tag{12}$$

where c is an absolute constant and $C(n) \to \infty$ arbitrarily slowly. Then a.e. graph satisfies

$$K' \leqslant d_m \leqslant K''.$$

This theorem tells us that in order to find a small interval to which d_m is confined in a.e. graph, we have to overcome two different problems. We have to determine, if possible, compute, K as a function of m (and n, of course) such that (10) holds, and then we have to find K' and K'', say in the form $K' = K - \varepsilon(pqn)^{1/2}$ and $K'' = K + \varepsilon(pqn)^{1/2}$, such that (11) and (12) hold. Let us see first how this ε can be chosen.

Lemma 9. *Let $C > 2$ be a constant and suppose $K = pn + x(pqn)^{1/2} = pn + o(n^{2/3})$ is such that $\mu = E(X_K) \to \infty$. Suppose furthermore that $C(n) \leqslant \mu^{1/2}$ and $C(n) \to \infty$. Then inequalities (11) and (12) of Theorem 8 are satisfied for some constant $c > 0$ if $K' = K - \varepsilon(pqn)^{1/2}$ and $K'' = K + \varepsilon(pqn)^{1/2}$ are chosen as follows.*

(a) *If $x \geqslant C$, then $\varepsilon = C(n)/(x\mu^{1/2})$.*

(b) *If $|x| \leqslant C$, then $\varepsilon = C(n)n^{-1/2}$.*

Proof. (a) Inequalities (1) and (3) imply that for sufficiently large n

$$\frac{1}{2\pi}\frac{n}{x} e^{-x^2/2} \leqslant c\frac{n}{x} e^{-x^2/2}.$$

This shows, in particular, that $\varepsilon(pqn)^{1/2} \to \infty$. We can use either (1′) and (4) or simply (2) to estimate $\mu' - \mu$. Indeed,

$$\mu' - \mu = nS(n-1); \lceil K' \rceil, \lceil K \rceil - 1),$$

so

$$n(\lceil K\rceil - \lceil K'\rceil)b(n-1: \lfloor K\rfloor) \leq \mu' - \mu \leq n(\lceil K\rceil - \lceil K'\rceil)b(n-1; \lceil K'\rceil).$$

Consequently (4) gives

$$\mu' - \mu \leq n\varepsilon(pqn)^{1/2}(pqn)^{-1/2}e^{-(x-\varepsilon)^2/2}$$

$$\leq n\varepsilon e^{-x^2/2}e^{x\varepsilon} \leq (2e)\frac{n}{x}e^{-x^2/2} \leq (4\pi e)\mu.$$

On the other hand, again from (4),

$$\mu' - \mu \geq n\varepsilon(pqn)^{1/2}(3\pi pqn)^{-1/2}e^{-x^2/2} \geq \tfrac{1}{4}n\frac{c(n)}{x\mu^{1/2}}e^{-1/2x^2}$$

$$\geq \frac{1}{4}\frac{C(n)}{\mu^{1/2}}\mu = \tfrac{1}{4}C(n)\mu^{1/2}.$$

This proves (11). Inequality (12) is proved analogously.

(b) A rather trivial consequence of (1) and (3) is that for $|x| \leq C$ we have

$$0 < c_1 n < \mu'' < \mu < \mu' < c_2 n.$$

Furthermore, equally trivially,

$$c_3 n^{-1/2} \leq b(n-1; k) \leq c_4 n^{-1/2}.$$

if $\lceil K'\rceil \leq k \leq \lceil K''\rceil$. Consequently

$$\left(\frac{pq}{c_2}\right)^{1/2} c_3 C(n)\mu^{1/2} \leq (\lceil K\rceil - \lceil K'\rceil]c_3 n^{1/2} \leq \mu' - \mu \leq c_2 n \leq \frac{c_2}{c_1}\mu,$$

proving (11), and (12) again follows analogously.

How shall we go about finding a K satisfying (10)? In a sense our job is very easy: we do know from (8) that there *is* such a K. However, in order to find such a K we have no choice but to look at the next two steps in the following sequence of approximations:

$$m \sim \mu = nS(n-1; K) \sim n(1-\Phi(x)) \sim \frac{1}{\sqrt{(2\pi)}}\frac{1}{x}e^{-x^2/2}.$$

Of course, we are not satisfied with the *ratios* being near to 1, we must bound the *differences* by $cm^{1/2}$ for some constant c. The first approximation, that is (3), is very good indeed. Theorem 2 of Littlewood [15, p. 53] shows that if $K = pn + x(pqn)^{1/2}$ and $x > C > 0$ then

$$|\mu - n(1-\Phi(x))| = o(x^3 n^{-1/2}\mu) = O(\mu^{1/2}).$$

(In fact, if $x \to \infty$, then the error term is clearly $o(\mu^{1/2})$.) Also, if $|x| \leq C$, then

$$|\mu - n(1-\Phi(x))| = O(n^{1/2}) = O(\mu^{1/2}).$$

Using only this first approximation we thus arrive at the main result of the paper.

Theorem 10. *Let $m \leq n/2$ be a natural number and let $C(n) \to \infty$ arbitrarily slowly. Define x by*

$$1 - \Phi(x) = m/n.$$

Then a.e. graph satisfies

$$|d_m - pn - x(pqn)^{1/2}| \leq C(n)\left(\frac{n}{m \log(n/m)}\right).$$

Proof. Because of Corollary 7 we may assume that $m \to \infty$. Putting together Theorem 8, Lemma 9 and the remarks above, we find that a.e. graph satisfies

$$|d_m - pn - x(pqn)^{1/2}| \leq \varepsilon(pqn)^{1/2},$$

where

$$\varepsilon = C(n)/(x\mu^{1/2}) \quad \text{if } x > c_1,$$

and

$$\varepsilon = C(n)n^{-1/2} \quad \text{if } 0 \leq x \leq c_1.$$

Now if $x \geq 4$, say, then $x \geq (\log(n/m))^{1/2}$ and $\mu^{1/2} > \frac{1}{2}m^{1/2}$ if n is large. Hence in this case

$$|d_m - pn - x(pqn)^{1/2}| \leq C(n)\left(\frac{n}{m \log(n/m)}\right)^{1/2},$$

as claimed. If, on the other hand, $x \leq 4$ then $\log(n/m) \leq 12$, hence

$$|d_m - pn - x(pqn)^{1/2}| \leq c_2 C(n)\left(\frac{n}{m \log(n/m)}\right)^{1/2}.$$

Incorporating c_2 into $C(n)$, as we may, we arrive at the required inequality. □

Theorem 10 shows the existence of a degree to which d_m is near in a.e. graph. Of course, we would prefer to have a formula that tells us the actual value, at least for large x. In order to find this value we have to examine the second approximation:

$$\mu_1 = n(1 - \Phi(x)) \sim \frac{1}{\sqrt{(2\pi)}}\frac{n}{x}\,\mathrm{e}^{-x^2/2} = \mu_2.$$

This approximation is given by (1) and it is not as good as it seems. However, it is accurate enough to give us (10) if $m = O((\log n)^s)$ for some fixed s, and we can use (1′) instead of (1) if $m = O((\log n)^2)$. Furthermore, if $m = O(\log n/(\log\log n)^4)$ then the approximation (9′) is accurate enough, so we may put $K_f = K_{\log m}$, where K_f is given by (9). Finally, it is clear that for $m - n/2 = O(n^{1/2})$ we may put $x = 0$ and $K = pn$. Let us summarize these remarks.

Corollary 11. *Let $C(n)$ be a function tending to ∞ as slowly as we like.*

(i) *If $m=O(\log n/(\log\log n)^4)$ then a.e. graph satisfies*

$$\left|d_m-pn-(2pqn\log n)^{1/2}+\log\log n\left(\frac{pqn}{8\log n}\right)^{1/2}\right.$$
$$\left.+\log(2\pi^{1/2}m)\left(\frac{pqn}{2\log n}\right)^{1/2}\right|\leq c(n)\left(\frac{n}{m\log n}\right)^{1/2}.$$

(ii) *If $m=O((\log n)^2)$, then define x by*

$$\frac{1}{\sqrt{(2\pi)}}\frac{1}{x}e^{-x^2/2}=\frac{\varphi(x)}{x}=\frac{m}{n}.$$

Then a.e. graph satisfies

$$|d_m-pn-x(pqn)^{1/2}|\leq C(n)\left(\frac{n}{m\log n}\right)^{1/2}.$$

(iii) *If $m-n/2=O(n^{1/2})$, then a.e. graph satisfies*

$$|d_m-pn|\leq(n).$$

Let us remark here that there is, of course, no need to go beyond $\frac{1}{2}n$ with m since we may interchange p and q and then our results tell us about d_m for $m\geq\frac{1}{2}n$. In particular, we see that a.e. degree (that is every degree with the exception of $o(n)$) is confined to an interval of length $C(n)n^{1/2}$ about pn. (This simple statement does not rely on the essential part of Theorem 10 and we could have deduced it from the trivial Lemma 2 above.)

As a final act of defiance, we sacrifice the short length of the interval to which d_m is confined for the location of this interval, and note the concrete bounds that can be read out of Theorem 10.

Theorem 12. *Suppose $m\to\infty$ and $m=o(n)$. Put*

$$\begin{aligned}K&=K(m,n)\\&=pn+(2pqn\log(n/m))^{1/2}-(\log\log(n/m)\\&\quad+\log 4\pi)\left(\frac{pqn}{8\log(n/m)}\right)^{1/2}.\end{aligned}$$

Then a.e. graph satisfies

$$|d_m-K|=O\left(\frac{n}{\log(n/m)}\right)^{1/2}.$$

Proof. Let $\varepsilon>0$. Define x_0 by $K=pn+x_0(pqn)^{1/2}$. Put $x_1=x_0-\varepsilon/(\log(n/m))^{1/2}$ and $x_2=x_0+\varepsilon/(\log(n/m))^{1/2}$. Furthermore, as in Theorem 10, denote by x the solution of

$$1-\Phi(x)=m/n.$$

Elementary but cumbersome calculations show that

$$\frac{1}{\sqrt{(2\pi)}}\frac{1}{x_1}e^{-x_1^2/2}<\frac{m}{n}$$

if n is sufficiently large. Recalling (1) we see that this implies that

$$1-\Phi(x_1)<m/n=1-\Phi(x)$$

and so $x_1<x$. Similarly

$$\frac{1}{\sqrt{(2\pi)}}\frac{1}{x_2}\left(1-\frac{1}{x_2^2}\right)e^{-x_2^2/2}>\frac{m}{n},$$

and this gives us $x<x_2$. Since $\varepsilon(pqn)^{1/2}\to\infty$, we can put $C(n)=\varepsilon(pqn)^{1/2}$ in Theorem 10 and obtain that a.e. graph satisfies

$$\begin{aligned}d_m&\leqslant pn+x(pqn)^{1/2}+C(n)\left(\frac{n}{m\log(n/m)}\right)^{1/2}\\&\leqslant pn+x_2(pqn)^{1/2}+\varepsilon(pqn)^{1/2}\left(\frac{n}{\log(n/m)}\right)^{1/2}\\&=pn+x_0(pqn)^{1/2}+2\varepsilon\left(\frac{pqn}{\log(n/m)}\right)^{1/2}=K+2\varepsilon\left(\frac{pqn}{\log(n/m)}\right)^{1/2}\end{aligned}$$

A lower bound is obtained analogously, completing our proof. □

3. Jumps and repeated values

The intervals that we have shown will contain two consecutive degrees, say d_m and d_{m+1} which overlap greatly. Thus our results so far do not even imply the result of Erdős and Wilson [12], that a.e. graph has a unique maximal degree, that is $d_1>d_2$. However, it turns out that Theorem 12 enables us to estimate the jumps d_m-d_{m+1} very easily indeed.

Theorem 13. *Let $m=o(n)$ and let $c(n)$ be a positive function tending to 0 as slowly as we like. Then a.e. graph has the property that*

$$d_1-d_{i+1}\geqslant\frac{c(n)}{m^2}\left(\frac{n}{\log(n/m)}\right)^{1/2}\quad\text{for every } i<m.$$

Proof. As in Theorem 10, let x be defined by

$$1-\Phi(x)=m/n.$$

Putting $C(n)=(pqm)^{1/2}$, we see that a.e. graph satisfies

$$d_m \geqslant pn + x(pqn)^{1/2} - (\log (n/m))^{-1/2}(pqn)^{1/2}$$
$$= pn + (x-\varepsilon)(pqn)^{1/2}$$

where

$$\varepsilon = (\log (n/m))^{-1/2}.$$

Thus our theorem will be proved if we show that almost no graph contains an ordered pair of vertices (u, v) such that

$$K = \lceil pn + (x-\varepsilon)(pqn)^{1/2} \rceil \leqslant d(u) \leqslant d(v) \leqslant d(u) + J - 1,$$

where

$$J = \left\lceil \frac{c(n)}{m^2} \left(\frac{n}{\log (n/m)} \right)^{1/2} \right\rceil.$$

Denote by $Z = Z(K, J)$ the number of such pairs. All we have to do is show that $E(Z) \to 0$ for then Chebyshev's inequality (5) completes the proof. Clearly

$$E(Z) = n(n-1)\left\{ p \sum_{k=K-1}^{n-2} b(n-2; k) \sum_{l=k}^{k+J-1} b(n-2; l) \right.$$
$$\left. + q \sum_{k=K}^{n-2} b(n-2; k) \sum_{l=k}^{k+J-1} b(n-2; l) \right\}$$
$$\leqslant n^2 \sum_{k=K-1}^{n-2} b(n-2; k) J b(n-2; k-1)$$
$$= n^2 J S(n-2; K-1) b(n-2; K-1)$$

since

$$b(n-2; k) \geqslant b(n-2; l) \quad \text{if } l \geqslant k \geqslant K-1.$$

Using (2) to estimate $b(n-2; K-1)$ and (1′) and (3) to estimate $S(n-2; K-1)$, we find that if n is large enough then

$$E(Z) \leqslant cn^2 J \frac{1}{x} e^{-(x-\varepsilon)^2/2} \frac{1}{n^{1/2}} e^{-(x-\varepsilon)^2/2}$$

where c is an absolute constant. Making use of the fact that $x \sim (2 \log (n/m))^{1/2}$ and so $x\varepsilon < 2$, and that

$$\frac{1}{\sqrt{(2\pi)}} \frac{1}{x} e^{-x^2/2} \sim \frac{m}{n},$$

we arrive at the inequality

$$E(Z) \leqslant c' n^{3/2} J x \left(\frac{m}{n} \right)^2 \leqslant c'' c(n),$$

where c' and c'' are absolute constants. Thus $E(Z) \to 0$ as required. □

Our next result shows that Theorem 13 is best possible.

Theorem 14. *Suppose $m \to \infty$ and $m = o(n)$. Let $C(n) \to \infty$ arbitrarily slowly. Then a.e. graph is such that*

$$d_i - d_{i+1} \leqslant \frac{C(n)}{m}\left(\frac{n}{\log(n/m)}\right)^{1/2} \quad \textit{for some } i < m.$$

Proof. Let us assume, as we may, that $C(n) < m$. Put

$$J = \left\lfloor \frac{C(n)}{m^2}\left(\frac{n}{\log(n/m)}\right)^{1/2} \right\rfloor,$$

$$\delta = (\log(n/m))^{-1/2}$$

and

$$\varepsilon = \frac{C(n)}{m^2}(\log(n/m))^{-1/2}.$$

It is easily checked that if n is large enough,

$$e^{-(x+2\delta+\varepsilon)^2/2} \geqslant c e^{-x^2/2} \tag{13}$$

and

$$\frac{1}{x+\delta} e^{-(x+\delta)^2/2} - \frac{1}{x+2\delta} e^{-(x+2\delta)^2/2} \geqslant \frac{c}{x} e^{-x^2/2} \tag{14}$$

where $c > 0$ is an absolute constant. Define x by

$$\frac{\varphi(x)}{x} = \frac{1}{\sqrt{(2\pi)}}\frac{1}{x} e^{-x^2/2} = \frac{m}{n}.$$

Then the proof of Theorem 12 shows that for a.e. graph

$$d_m < K = \lceil pn + (x+\delta)(pqn)^{1/2} \rceil.$$

Consequently the assertion of our theorem will follow if we prove that $P(Z > 0) \to 1$, where $Z = Z(K, J)$ is, as above, the number of ordered pairs of vertices (x, y) satisfying $K \leqslant d(x) \leqslant d(y) \leqslant d(x) + J$. The above expression for $E(Z)$ shows immediately that for $K' = \lfloor pn + (x+2\delta)(pqn)^{1/2} \rfloor$ we have

$$\mu = E(Z) > \tfrac{1}{2}n^2(J+1)b(n, K'+J)S(n, K, K').$$

Making use of (1′), (2), (3), (13) and (14) to estimate the terms on the right-hand side, we find that

$$\mu \geqslant c' n^2 \frac{C(n)}{m^2}\left(\frac{n}{\log(n/m)}\right)^{1/2} \frac{1}{n^{1/2}} e^{-x^2/2} \frac{1}{x} e^{-x^2/2},$$

where $c' > 0$ is an absolute constant. Recalling the definition of x, this gives us

$$\mu \geqslant \frac{c'}{2\pi} C(n)\left(\frac{1}{\log(n/m)}\right)^{1/2} x.$$

As $x \sim (2\log(n/m))^{1/2}$, we see that $\mu \to \infty$.

It is straightforward to check that

$$\sigma^2 = E(|Z-\mu|^2) = E(Z^2) - \mu^2 = o(\mu^2).$$

Consequently, by Chebyshev's inequality (6′)

$$P(Z>0) \to 1$$

as required. □

Theorems 13 and 14 tell us rather precisely how many terms of the sequence $d_1 \geq d_2 \geq \cdots$ are strictly decreasing.

Corollary 15. *If $m = o(n^{1/4})/(\log n)^{1/4}$ then a.e. graph is such that $d_1 > d_2 > \cdots > d_m$; if $m \neq o(n^{1/4})/(\log n)^{1/4}$ then a.e. graph is such that $d_i = d_{i+1}$ for some $i < m$.*

Our next aim is to determine those values $k \geq p(n-1)$ for which a.e. graph contains a vertex of degree k. Furthermore, we shall give bounds on the number of vertices of degree k in a.e. graph. Note that the change described in Corollary 15 happens around degree $\lfloor pn + (\frac{3}{2}pqn\log n)^{1/2} \rfloor$.

Theorem 16. *Let $p(n-1) \leq k = p(n-1) + x(pqn)^{1/2} = p(n-1) + (pqn\log n)^{1/2} + y(pqn/\log n)^{1/2}$. If $y \to \infty$ then almost no graph contains a vertex of degree k and if $y \to -\infty$ then a.e. graph contains a vertex of degree k. Furthermore, if M denotes the multiplicity of k and $y \to -\infty$, then a.e. graph satisfies*

$$|M-\mu| \leq C(n)\mu^{1/2},$$

where

$$\mu = n\binom{n-1}{k}p^k q^{n-1-k},$$

and $C(n)$ is any function tending to ∞.

In particular, a.e. graph contains at least $n^{1/2}/(2\pi pq)^{1/2} - C(n)m^{1/4}$ vertices of degree exactly $k = \lfloor p(n-1) \rfloor$.

Proof. The expectation of the random variable M is clearly

$$E(M) = nb(n-1; k) = \mu.$$

Recalling (2) and (5) this implies immediately that if $y \geq C > 0$ then for large n

$$\mu \leq nn^{-1/2}e^{-(\log n)/2}e^{-C} = e^{-C}$$

and so $y \to \infty$ implies

$$P(M \geq 1) \leq \mu \to 0.$$

Assume now that $y \to -\infty$. Then by (2)

$$\mu \sim \frac{n^{1/2}}{\sqrt{(2\pi pq)}} e^{-x^2/2} \to \infty.$$

Exactly as in Lemma 3, one can check that

$$\sigma^2 = E(|M-\mu|^2) \leq 2\mu.$$

The calculations are particularly pleasing in the special case $p = q = \frac{1}{2}$, $n = 2k+1$, and give an even sharper estimate:

$$\begin{aligned} E(M^2) &= \mu + n(n-1)\{\tfrac{1}{2}b(2k-1;k)^2 + \tfrac{1}{2}b(2k-1;k-1)^2\} \\ &= \mu + n(n-1)b(2k-1;k-1)^2 = \mu + n(n-1)b(2k;k)^2 \\ &< \mu + \mu^2, \end{aligned}$$

so

$$\sigma^2 < \mu.$$

Consequently by (6) for every $c > 0$ we have

$$P(|M-\mu| \geq c\mu^{1/2}) \leq 2/c^2.$$

The last assertion follows if we replace μ by its sufficiently accurate estimate. □

4. Three applications

In this section we shall apply our main results to estimate the sum of the m highest degrees, the maximal number of edges covered by m vertices and the vertex connectivity of large subgraphs. In order to simplify the formulae and the calculations we shall not strive for best possible results.

Theorem 17. *Let $\varepsilon > 0$ and suppose $m \to \infty$ and $m \leq n^{1-\varepsilon}$. Then for a.e. graph*

$$S_m = \sum_1^m d_i = mpn + m(2pqn \log(n/m))^{1/2} + o(m(n \log n)^{1/2}).$$

Proof. Since

$$S_m = \sum_1^m d_i \geq md_m,$$

Theorem 12 implies immediately that S_m is at least as large as claimed. Let now $0 < \delta < 1$ and put $m' = \lfloor m^{1-\delta} \rfloor$. Then, again by Theorem 12, for a.e. graph

$$\begin{aligned} d_{m'} &\leq pn + (2pqn(\log(n/m) + \delta \log m))^{1/2} \\ &\leq pn + (2pqn \log(n/m))^{1/2} + \frac{\delta}{2\varepsilon^{1/2}} \frac{\log m}{(\log(n/m))^{1/2}} (2pqn)^{1/2} \end{aligned}$$

and

$$d_1 \leq pn + (2pqn \log n)^{1/2}.$$

Hence

$$S_m \leq (m-m')d_{m'} + m'd_1 \leq mpn + m(2pqn \log (n/m))^{1/2}$$
$$+ m\frac{\delta}{2\varepsilon^{1/2}} \frac{\log m}{(\log (n/m))^{1/2}} (2pqn)^{1/2} + m'(2pqn \log n)^{1/2}.$$

Since δ is as small as we like, the last two terms can be replaced by $o(m(n \log n)^{1/2})$, and we are home. □

Let us investigate now the maximal number of edges that can be covered by m vertices. The following lemma will enable us to use Theorem 17 to estimate this number.

Lemma 18. *Let $\varepsilon > 0$. Then a.e. graph is such that whenever $m > n^{\varepsilon}$ every subgraph of order m has $p\binom{m}{2} + o(m^2)$ edges.*

Proof. It is a trivial consequence of Theorem 2 of [15, p. 53] that if $0 < \delta < \frac{1}{8}$ and N is sufficiently large then

$$\sum_{|k-pN| \geq \delta N} b(N; K) \leq e^{-\delta^2 N}.$$

This inequality enables us to estimate the expected number of subgraphs of order m and size M with $|M - p\binom{m}{2}| \geq \delta\binom{m}{2}$. Putting $N = \binom{m}{2}$ we have

$$\binom{n}{m}\left\{\sum_{|k-pN| \geq \delta N} b(N; k)\right\} \leq n^m e^{-\delta^2 m^2/2} < n^{-2}$$

if n is sufficiently large (depending only on ε and δ). Consequently the expected number of subgraphs of any order $m > n^{\varepsilon}$ is at most $1/n$. □

Theorem 19. *Let $0 < \varepsilon < \frac{1}{2}$ and let m be such that $n^{\varepsilon} < m < n^{1-\varepsilon}$. Denote by W_m the maximal number of edges that can be covered with m vertices. Then a.e. graph is such that*

$$W_m = mpn + m(2pqn \log (n/m))^{1/2} - p\binom{m}{2} + o\{m^2 + m(n \log (n/m))^{1/2}\}.$$

Proof. As in Theorem 17, denote by S_m the sum of the m largest degrees. Let e_* be the minimal number of edges in a graph of order m and let e^* by the maximal number. Then clearly

$$S_m - e^* \leq W_m \leq S_m - e_*,$$

so our assertion follows from Lemma 18 and Theorem 17. □

As a.e. graph has diameter 2, we know from a result in [3] that the minimum degree equals the edge connectivity in a.e. graph. Our last result is a considerable extension of this trivial statement. We shall show that a.e. graph is such that for the large subgraphs the minimum degree equals the vertex connectivity and the omission of certain vertices increases the vertex connectivity considerably. In the proof of our result we shall need the following simple lemma.

Lemma 20. *Let s be a fixed natural number and let $\varepsilon > 0$. Then a.e. graph is such that whenever $x_1, x_2, \dots, x_r$ are distinct vertices and $r \leqslant s$,*

$$\left|\left|\bigcap_{i=1}^{r} \Gamma(x_i)\right| - p^r n\right| + \left|\left|\bigcup_{i=1}^{r} \Gamma(x_i)\right| - (1-q^r)n\right| < \varepsilon n,$$

where $\Gamma(x_i)$ denotes the set of neighbours of x_i.

Theorem 21. *Almost every graph G contains $t = \lfloor n^{1/7} \rfloor$ distinct vertices $g_1, g_2, \dots, g_t$ such that if $G_0 = G$ and $g_i = G_{i-1} - \{y_i\}$, $i = 1, 2, \dots, t$, then*

$$\kappa(G) < \kappa(G_1) - t < \kappa(G_2) - 2t < \cdots < \kappa(G_t) - t^2.$$

Furthermore, for each i, $0 \leqslant i \leqslant t$, we have $\kappa(G_i) = \delta(G_i)$ and G_i has larger vertex-connectivity than any other subgraph of order $n - i$.

Proof. (a) Let $p < p' < 2p - p^2$. Then a.e. graph has maximal degree less than $p'n - t$.

(b) Because of Lemma 20 we may choose s so that in a.e. graph any s vertices are joined to more than $n(1+p')/2$ vertices and any s vertices are joined to more than $p'n + s$ vertices.

(c) By applying Theorem 13 to the complement of our graph, we see that in a.e. graph the tail end of the degree sequence $\delta = d_n = d(y_1) \leqslant d_{n-1} = d(y_2) \leqslant \cdots$, is such that

$$d(y_{i+1}) - d(y_i) > 2t \qquad \text{if } i \leqslant t.$$

We claim that if G has the properties described in (a), (b) and (c), then the vertices $y_1, y_2, \dots, y_t$ satisfy the conditions.

Let H be a subgraph of order $n - i$, $0 \leqslant i \leqslant t$. Suppose $y_h \in H$ but $y_j \notin H$ for $j < h$. Of course, $h \leqslant i + 1$. If $y \in H$ and $y \neq y_h$ then

$$d_H(y_h) \leqslant d(y_h) < d(y) - 2t < d_H(y) - t.$$

This shows that the minimal degree of G_i is exactly $d_{G_i}(y_{i+1})$ and it is larger than the minimal degree of any other subgraph of order $n - i$. Furthermore, $\delta(G_i) < \delta(G_{i+1}) - t$.

All that remains to be proved is that $\kappa(G_i) = \delta(G_i)$, $1 \leqslant i \leqslant t$. If this is not the case for some G_i then G_i contains a set W of $\kappa(G_i) \leqslant \Delta(G_i) \leqslant \Delta(G) < p'n - t$ vertices such that $G_1 - W = H_1 \cup H_2$ and $2 \leqslant |H_1| \leqslant |H_2|$, say. By (b) any two vertices in H_1 are joined to at least $p'n+s$ vertices. Hence

$$|H_1|+|W|+t \geqslant p'n+s$$

and so

$$|H_1| \geqslant s.$$

Again by (b) any s vertices of H_1 are joined to more than $\frac{1}{2}n(1+p')$ vertices and so

$$\tfrac{1}{2}n(1+p') < |H_1|+|W|+t \leqslant \tfrac{1}{2}(n+|W|+t) < \tfrac{1}{2}n(1+p').$$

This contradiction completes the proof. □

References

[1] B. Bollobás, Extremal Graph Theory (Academic Press, London, 1978).
[2] B. Bollobás, Chromatic number, girth and maximum degree, Discrete Math. 24 (1978) 311–314.
[3] B. Bollobás, Graphs with equal minimum degree and edge connectivity, Discrete Math. 28 (1979) 321–323.
[4] B. Bollobás and P. Erdös, Cliques in random graphs, Proc. Cambridge Philos. Soc. 80 (1976) 419–427.
[5] B. Bollobás and N. Sauer, Uniquely colourable graphs with large girth, Canad. J. Math. 28 (1976) 1340–1344.
[6] B. Bollobás and A.G. Thomason, Uniquely partitionable graphs, J. London Math. Soc. (2) 16 (1977) 403–410.
[7] P. Erdös, Graph theory and probability, Canad. J. Math. 11 (1959) 34–38.
[8] P. Erdös, Graph theory and probability II, Canad. J. Math. 13 (1961) 346–352.
[9] P. Erdös and A. Rényi, On random graphs I, Publ. Math. Debrecen 6 (1959) 290–291.
[10] P. Erdös and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hungar. Acad. Sci. 5 (1960) 17–61.
[11] P. Erdös and J. Spencer, Probability Methods in Combinatorics (Academic Press, New York, 1974).
[12] P. Erdös and R. J. Wilson, On the chromatic index of almost all graphs, J. Combinatorial Theory Ser. B 23 (1977) 255–257.
[13] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 1, 2nd ed. (John Wiley and Sons, New York, 1975).
[14] G.R. Grimmett and C.J.H. McDiarmid, On colouring random graphs, Proc. Cambridge Philos. Soc. 77 (1975) 313–324.
[15] J.E. Littlewood, On the probability in the tail of the binomial distribution, Advances in Appl. Probability 1 (1979) 43–72.
[16] D.W. Matula, On the complete subgraphs of a random graph, in: Combinatory Mathematics and Its Applications (Univ. of North Carolina Press, Chapel Hill, N.C., 1970) 356–369.
[17] D.W. Matula, The employee party problem, Preprint.
[18] L. Pósa, Hamiltonian cycles in random graphs, Discrete Math. 14 (1976) 359–364.
[19] A. Rényi, Foundations of Probability (Holden-Day, San Francisco, 1970).
[20] N. Sauer and J. Spencer, Edge disjoint placements of graphs, J. Combinatorial Theory Ser. B.
[21] J. Spencer, Turán's theorem for k-graphs, Discrete Math. 2 (1972) 183–186.